Problem of Erlang B
Assume infinite users.
λ_u

Engset

4 Engsett Model

In this section, we will consider a more realistic system where the infinite-user assumption in M/M/m/m is relaxed.

4.1. Modified (more realistic) assumptions:

(a) Finite number of users: n users/customers

(b) Each user generates new call request with rate λ_u

- Total call request rate $= n \times \lambda_u = \lambda$

4.2. We need to modify our small slot analysis. Here, for each small slot, each user do the following:

(a) If it is idle (not using the channel) at the beginning of the slot,

(i) it may request/generate a new call in a small slot with probability $\lambda_u \delta$.

i. If there is at least one available channel, then it may start its conversation. (In which case, at the beginning of the next slot, its call is ongoing.)

ii. If there is no channel available, then the call is blocked and it is idle again (at the beginning of the next slot).

(ii) With probability $1 - \lambda_u \delta$, no new call is requested by this user during this slot. (In which case, it is idle at the beginning of the next slot.

(b) If it is making a call at the beginning of the slot,

(i) the call may end with probability $\mu\delta$. (In which case, at the beginning of the next slot, it is idle.)

(ii) With probability $1 - \mu\delta$, the call is still ongoing at the end of this slot (which is the same as the beginning of the next slot.)

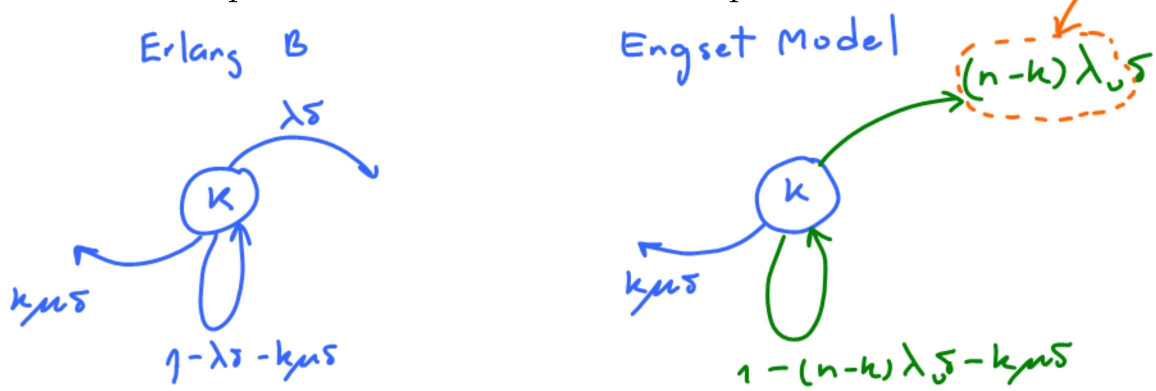
4.3. Observe that the call generation process for each user is not a Poisson process with rate λ_u. This is because it get interleaved with the call duration for each successful call request. Part of the Poisson assumption that is left is that, in fact, for an idle user, the time until the next call request will be exponential with rate λ_u.

users

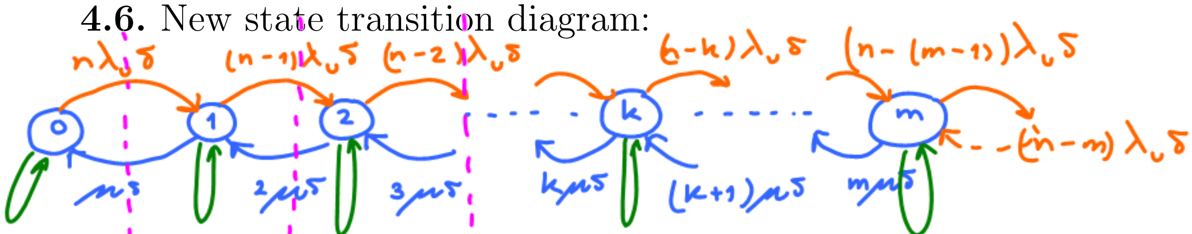
4.4. A user can not generate new call when he/she is already involved in a call. Therefore, if the system is in state $K = k$, there are only $n - k$ users that can generate new calls. Hence, the "total" call request rate for state $K = k$ is $(n - k)\lambda_u$.

- Earlier, when we consider the Erlang B formula, we always have λ as the total new call request rate regardless of how many users are using the channels. This is because we assumed infinite number of users and hence having k users using the channels will not significantly change the total call request rate.

4.5. Comparison of the state transition probabilities:



4.6. New state transition diagram:



$p_0 n\lambda_u \cancel{\delta} = p_1 \mu \cancel{\delta}$

$p_1 = nA_u p_0$

$p_1(n-1)\lambda_u \cancel{\delta} = p_2 2\mu \cancel{\delta}$

$p_2 = \frac{n-1}{2} A_u p_1$

$= \frac{(n-1)}{2} n A_u^2 p_0$

$= \binom{n}{2} A_u^2 p_0$

$p_2 (n-2)\lambda_u \cancel{\delta} = p_3 3\mu \cancel{\delta}$

$p_3 = \frac{n-2}{3} A_u p_2$

$= \frac{n(n-1)(n-2)}{3\times 2} A_u^3 p_0$

$= \binom{n}{3} A_u^3 p_0$

$p_k = \binom{n}{k} A_u^k p_0$

$= \frac{\binom{n}{k} A_u^k}{\sum_{i=0}^{m} \binom{n}{i} A_u^i} = \frac{\binom{n}{k} A_u^k}{z(m,n)}$

$p_0 + p_1 + \cdots + p_m = 1$

$\sum_{k=0}^{m} p_k = 1$

$\sum_{k=0}^{m} \binom{n}{k} A_u^k p_0 = 1$

$p_0 = \frac{1}{\underbrace{\sum_{k=0}^{m} \binom{n}{k} A_u^k}_{z(m,n)}}$

Truncated Poisson pmf.

Truncated binomial pmf.

4.7. Comparison of the steady-state probabilities:

Erlang B

$$p_k = \frac{A^k/k!}{\sum_{i=0}^{m} \frac{A^i}{i!}}$$

∞ in Poisson

p_m = call blocking probability

Engset

$$p_k = \frac{\binom{n}{k} A_u^k}{\sum_{i=0}^{m} \binom{n}{i} A_u^i}$$

= n in Binomial

$p_m \neq$ call blocking probability.

4.8. It is tempting to conclude that the call blocking probability is p_m. ← wrong
However, this is not the case for us here. Recall that p_m is the long-run probability (and the long-run proportion of time) that the system will be in state $K = m$. In this state, any new call request will be blocked. So, p_m gives the blocking probability in terms of the ***time*** (time congestion).

However, if you look back at how we define P_b which is the ***call*** blocking probability, this is the probability that a ***call*** is blocked. So, what we need to find out is, out of all the new calls that are requested, how many will be blocked.

To do this, consider s slots. Here the value of s is very large. Then,...

(a) About $p_k \times s$ slots will be in state k.

$\approx p_0 \times s$ will be in state 0

$\approx p_1 \times s$ will be in state 1

(b) Each of these slots will generate new call request with probability $(n - k)\lambda_u\delta$.

depending on the state it is in

(c) So, the number of new calls request from slots that are in state k will be approximately

This color for Erlang B

$$\overbrace{(n-k)\lambda_u\delta}^{\lambda} \times \overbrace{(p_k \times s)}^{\approx p_k \times s \text{ slots will be in state } k}.$$

(d) Therefore, total number of new call requests will be approximately

$$\sum_{k=0}^{m} \overbrace{(n-k)\lambda_u\delta}^{\lambda} \times (p_k \times s).$$

(e) However, the number of the new call requests that get blocked is

$$\overbrace{(n-m)\lambda_u\delta}^{\lambda} \times (p_m \times s).$$

(f) Hence, the proportion (probability) of calls that are blocked is

$$\frac{\overbrace{(n-m)\cancel{\lambda_u\delta}}^{\lambda} \times (p_m \times \cancel{s})}{\sum_{k=0}^{m} \underbrace{(n-k)\cancel{\lambda_u\delta}}_{\lambda} \times (p_k \times \cancel{s})} = \frac{(n-m)p_m}{\sum_{k=0}^{m}(n-k)p_k}.$$

Plugging in the values of p_k and p_m which we got earlier, we then get

$$\boxed{P_b = \frac{(n-m)p_m}{\sum\limits_{k=0}^{m}(n-k)p_k}} = \frac{(n-m)\frac{A_u^m\binom{n}{m}}{z(m,n)}}{\sum\limits_{k=0}^{m}(n-k)\frac{A_u^k\binom{n}{k}}{z(m,n)}} = \boxed{\frac{(n-m)A_u^m\binom{n}{m}}{\sum\limits_{k=0}^{m}(n-k)A_u^k\binom{n}{k}}}.$$

4.9. Comparison of the call blocking probability:

Erlang B

$$P_b = \frac{\cancel{\lambda} p_m}{\sum\limits_{k=0}^{m} \lambda p_k} = p_m$$

Engset

$$P_b = \frac{(n-m)p_m}{\sum\limits_{k=0}^{m}(n-k)p_k}$$

4.10. Remarks:

$\lambda_u = \frac{\lambda}{n}$

(a) If we keep the total rate λ constant and let $n \to \infty$, then the call blocking probability in the Engsett model will be the same as the call blocking probability in the Erlang model. [HW3]

\# channels

(b) If $n \le m$, the call blocking probability in Engsett model will be 0.

(c) M/M/m/m for Erlang B formula

Poisson arrival process

call duration process

In HW3, we will see M/M/m/∞ → Erlang C.

References

[1] Andrea Goldsmith. *Wireless Communications*. Cambridge University Press, 2005. 2

[2] J. R. Norris. *Markov Chains*. Cambridge University Press, 1998. 3.1

[3] Theodore S. Rappaport. *Wireless Communications: Principles and Practice*. Prentice Hall PTR, 2 edition, 2002. 3.8